Fundamental groups and presentations of algebras

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Abstract. In this note, we investigate how different fundamental groups of presentations of a fixed algebra A can be. For finitely many finitely presented groups G_i, we construct an algebra A such that all G_i appear as fundamental groups of presentations of A.

Introduction

Let A be a basic, connected, finite dimensional algebra over an algebraically closed field k. By a result of Gabriel [9], there exists a unique finite connected quiver Q and a two-sided ideal I of the path algebra kQ, such that $A \cong kQ/I$. Such a pair (Q, I) is called a bound quiver. The morphism $\nu : kQ \longrightarrow A \simeq kQ/I$, as well as (Q, I) are both called **presentations** of A. Following [14], one can define the fundamental group $\pi_1(Q, I)$. Moreover, by a result of Fischbacher and de la Peña [8], every finitely presented group arises in this way. An important feature of these groups is that they depend essentially on the ideal I, thus it is not an invariant of the algebra. Actually, there are known examples of algebras $A \cong kQ/I_1 \cong kQ/I_2$ such that $\pi_1(Q, I_1) \not\cong \pi_1(Q, I_2)$ (see example 1 in 1.2, and section 3).

With this in mind, we tackle the following question: Given an algebra A, as above, how distinct can be fundamental groups of presentations of A?

We consider essentially two settings, namely the triangular and the not triangular case. In the triangular case, we consider groups which are obtained from finite free products of finitely generated abelian groups. If we allow loops, we are able to obtain results concerning finitely generated groups.

This paper is organized as follows: In Section 1, we fix notations and terminology, recall definitions concerning fundamental groups of bound quivers and give some Examples. In Section 2, we deal with products (and coproducts) of bound quivers which yield to products (and coproducts) of their fundamental groups. Section 3 is devoted to investigate the effects of changes of presentations on products

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(and coproducts) and fundamental groups. Finally, in Section 4, we prove the main result:

Theorem A. *Let $G_1, \ldots, G_n$ be finitely presented groups. Then, there exists an algebra A having presentations $A \simeq kQ_A/I_i$, for $i \in \{1, \ldots, n\}$, such that $\pi_1(Q_A, I_i) \simeq G_i$.*

The quivers considered in Theorem A have loops, so they lead to algebras of infinite global dimension. Nevertheless, we obtain a weaker result concerning triangular algebras. In this setting, we consider the family of groups obtained from cyclic groups by performing finite free and direct products, which we denote $\mathbb{G}$.

Theorem B. *Let $G_1, \ldots, G_n \in \mathbb{G}$. Then, there exists a triangular algebra A having presentations $A \simeq kQ_A/I_i$, for $i \in \{1, \ldots, n\}$, such that $\pi_1(Q_A, I_i) \simeq G_i$.*

1. Preliminaries

1.1. Bound quivers and algebras. A quiver Q is a quadruple (Q_0, Q_1, s, t), where Q_0 and Q_1 are sets, and s, t maps $s, t : Q_1 \longrightarrow Q_0$. The elements of Q_0 are the **vertices** of Q, whereas the elements of Q_1 are its **arrows**. Given an arrow α in Q_1, the vertex $s(\alpha)$ is called its source and $t(\alpha)$ its target, and we write $\alpha : s(\alpha) \longrightarrow t(\alpha)$. A **path** w is a sequence of arrows $w = \alpha_1\alpha_2 \cdots \alpha_n$ such that for $i \in \{1, \ldots, n-1\}$, one has $t(\alpha_i) = s(\alpha_{i+1})$. The source and the terminus of a path w in Q are defined in the obvious way. A quiver Q is said to be **finite** if both, Q_0 and Q_1 are finite sets. We say that Q is **connected** if the underlying graph of Q is connected. Unless it is otherwise stated, we will consider finite and connected quivers.

Given a commutative field k and a quiver Q, the path algebra kQ is the $k-$vector space whose basis is the set of paths of Q, including one stationary path e_x for each vertex x of Q. The multiplication of two basis elements of kQ is their composition whenever it is possible, and 0 otherwise. Let F be the two-sided ideal of kQ generated by the arrows of Q. A two-sided ideal I of kQ is called **admissible** if there exists an integer $m \geq 2$ such that $F^m \subseteq I \subseteq F^2$. The pair (Q, I) is then called a **bound quiver**. Naturally, a **pointed bound quiver** (Q, I, x) is a bound quiver together with a distinguished vertex $x \in Q_0$.

Conversely, let A be a finite dimensional algebra over an algebraically closed field k. It is well-known (see [9, 4] for example) that, if in addition we assume that A is a basic and connected, then there exists a unique finite connected quiver Q_A and a surjective morphism of $k-$algebras $\nu : kQ_A \longrightarrow A$, which is not unique, with $I = \operatorname{Ker} \nu$ an admissible ideal. Those surjective morphisms, or equivalently the pairs (Q_A, I), are called **presentations** of the algebra A. Remark that a morphism $\nu : kQ_A \longrightarrow A$ is a presentation of A whenever $\{\nu(e_x) \mid x \in (Q_A)_0\}$ is a complete set of primitive orthogonal idempotents and, for any fixed $x, y \in (Q_A)_0$, we have that $\{\nu(\alpha) + \mathrm{rad}^2 A |\ \alpha : x \longrightarrow y \in Q_1\}$ is a basis of $\nu(e_x)(\mathrm{rad}A/\mathrm{rad}^2 A)\nu(e_y)$.

In this note, for a given bound quiver (Q, I), we will consider morphisms $\nu : kQ \longrightarrow kQ/I \cong A$ defined by $\nu(e_x) = e_x + I$ for $x \in Q_0$, and, given an arrow $\alpha \in Q_1$, from, say x to y, $\nu(\alpha) = \alpha + \rho_\alpha + I$ where ρ_α is a linear combination of paths from x to y different from α. In particular, if the paths appearing in ρ_α have length at least 2, then $\nu : kQ \longrightarrow A$ is a presentation of A.

We refer the reader to [2, 1], for instance, for further reference on the use of bound quivers in the representation theory of algebras.

1.2. Fundamental groups of bound quivers. Given a bound quiver (Q, I), its fundamental group is defined as follows (see [14]). For $x,\ y$ in Q_0, set $I(x, y) = e_x(kQ)e_y \cap I$. A relation $\rho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$ (where $\lambda_i \in k_*$, and w_i are different paths from x to y) is said to be **minimal** if $m \geq 2$, and, for every proper subset J of $\{1, \ldots, m\}$, we have $\sum_{i \in J} \lambda_i w_i \notin I(x, y)$. For a given arrow $\alpha : x \longrightarrow y$, let $\alpha^{-1} : y \longrightarrow x$ be its formal inverse. A **walk** w in Q from x to y is a composition $w = \alpha_1^{\epsilon_1} \alpha_2^{\epsilon_2} \cdots \alpha_n^{\epsilon_n}$ such that the $s(\alpha_1^{\epsilon_1}) = x,\ t(\alpha_n^{\epsilon_n}) = y$, and, for $i \in \{2, \ldots n\}$, $s(\alpha_i^{\epsilon_i}) = t(\alpha_{i-1}^{\epsilon_{i-1}})$. Define the **homotopy relation** $\sim$ on the set of walks on (Q, I), as the smallest equivalence relation satisfying the following conditions :

(1) For each arrow α from x to y, one has $\alpha\alpha^{-1} \sim e_x$ and $\alpha^{-1}\alpha \sim e_y$.
(2) For each minimal relation $\sum_{i=1}^{m} \lambda_i w_i$, one has $w_i \sim w_j$ for all i, j in $\{1, \ldots, m\}$.
(3) If u, v, w and w' are walks, and $u \sim v$ then $wuw' \sim wvw'$, whenever these compositions are defined.

We denote by $\widetilde{w}$ the homotopy class of a walk w. Let v_0 be a fixed point in Q_0, and consider the set $W(Q, v_0)$ of walks of source and target v_0. On this set, the product of walks is everywhere defined. Because of the first and the third conditions in the definition of the relation $\sim$, one can form the quotient group $W(Q, v_0)/\sim$. This group is called the **fundamental group** of the bound quiver (Q, I) with base point v_0, denoted by $\pi_1(Q, I, v_0)$. It follows easily from the connectedness of Q that this group does not depend on the base point v_0, and we denote it simply by $\pi_1(Q, I)$. This group has a clear geometrical interpretation as the first homotopy group of a C.W. complex $\mathcal{B}(Q, I)$ associated to (Q, I), see [6] (also [11]).

Remark. An important remark, which is the main motivation of this work, is that the group defined above depends essentially on the minimal relations of the ideal I. It is well-known that, for a $k-$algebra A, its presentation as a bound quiver algebra is not unique. Thus, the fundamental group is not an invariant of the algebra, as the following well-known example shows (see also section 3)

Example 1. Consider the quiver $3 \underset{\gamma}{\overset{\beta}{\rightrightarrows}} 2 \xrightarrow{\alpha} 1$ bound by the ideal $I_1 =< \beta\alpha >$. Since the ideal I_1 is generated by monomial relations, the homotopy relation is trivial, thus $\pi_1(Q, I_1) \simeq \mathbb{Z}$. On the other hand, consider the morphism of algebras $\nu_2 : kQ \longrightarrow A$ defined by $\nu_2(\beta) = (\beta + \gamma) + I_1$, and $\nu_2(\alpha) = \alpha + I_1$, $\nu_2(\gamma) = \gamma + I_1$. A straightforward computation shows that ν_2 is a presentation, and, moreover, that $I_2 = \text{Ker } \nu_2 =< (\beta - \gamma)\alpha >$. This yields to a trivial group $\pi_1(Q, I_2)$.

On the other hand, these groups are invariant for some classes of algebras, as the following theorem states. Recall that a $k-$algebra $A = kQ/I$ is said to be **constricted** if for every arrow $\alpha : x \longrightarrow y$ in Q one has $\dim_k e_x A e_y = 1$.

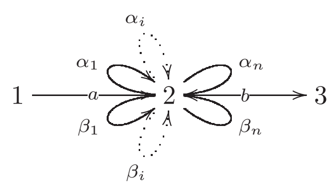
Theorem. (Bardzell - Marcos [3]) *Let $A = kQ/I$ be a constricted algebra. Given (Q, I_1), (Q, I_2), two presentations of A one then has $\pi_1(Q, I_1) \simeq \pi_1(Q, I_2)$.* □

It has been proved in [8] that given a finitely presented group G, there exists an incidence algebra $A = kQ/I$ such that $\pi_1(Q, I) \simeq G$. A natural question is then: *How distinct can be fundamental groups of presentations of an algebra?* Recalling that incidence algebras are always constricted (and triangular), and in light of the

Bardzell - Marcos theorem, this class of algebras is not interesting in view of our problem.

The following example shows how given any finitely presented group G, one can obtain a finite-dimensional non constricted nor triangular algebra $A \simeq kQ/I$, such that $\pi_1(Q, I) \simeq G$.

Example 2. Let $G =< \alpha_i |\ w_j, 1 \leq i \leq n, 1 \leq j \leq m >$ be a finitely presented group. More precisely, G is the factor group of the free group having basis $\{\alpha_i\}_{i=1}^n$ by the normal subgroup generated by $\{w_i\}_{i=1}^m$. Without loose of generality, we assume that the words w_j are reduced, non-empty, pairwise different, and, moreover, that each w_j contains at least one letter among the α_i's (otherwise we can replace w_j by w_j^{-1}). Consider the quiver Q_G:



Identifying the arrow β_i with the letter α_i^{-1}, we obtain a correspondence between the words in $\{\alpha_i, \alpha_i^{-1}\}_{i=1}^n$ and (some of) the paths of Q_G. With this identification in mind, define the ideal

$$I_G =< a\alpha_i\beta_i b - ab,\ aw_j b - ab, F^N | 1 \leq i \leq n, 1 \leq j \leq m >$$

where $N = \max\{l(w_j) + 3, 6 | 1 \leq j \leq m\}$. We claim that the generators of I_G which are not monomial relations are in fact minimal relations.

Indeed, if this is not the case, then we have $ab \in I_G$. That is, there are scalars $\lambda_i,\ \mu_j$ with $1 \leq i \leq n, 1 \leq j \leq m$ and an element in F^N such that

$$ab = \sum_{i=1}^{n} \lambda_i (a\alpha_1\beta_i b - ab) + \sum_{j=1}^{m} \mu_j (aw_j b - ab) + \gamma$$

and this is equivalent to

$$(\sum_{i=1}^{n} \lambda_i + \sum_{j=1}^{m} \mu_j - 1)ab + \sum_{i=1}^{n} \lambda_i a\alpha_i\beta_i b + \sum_{j=1}^{m} \mu_j aw_j b + \gamma = 0$$

But then, since $w_j \neq \alpha_i\beta_i$ for all i, j, and γ is a linear combination of paths of length $N \geq l(w_i) + 3 \geq 5$ we obtain $\lambda_i = \mu_j = 0$ for all i, j and $\gamma = 0$, which is a contradiction.

Therefore, $a\alpha_i\beta_i b \sim ab$, and $aw_j b \sim ab$. From this we get $\alpha_i\beta_i \sim e_2$, and $w_j \sim e_2$, for all i, j. This shows that $\pi_1(Q_G, I_G) \simeq G$. In section 3 we will consider changes of presentations of $A = kQ_G/I_G$.

If one wishes to avoid quivers having oriented cycles, but still have algebras that are not constricted, things are more difficult.

1.3. Coverings. Let $(\widehat{Q}, \widehat{I})$ be a (possibly infinite) bound quiver, and G a group of automorphisms acting freely on $(\widehat{Q}, \widehat{I})$. This ensures that we can form the quotient $(Q, I) = (\widehat{Q}, \widehat{I})/G$. The natural map $p : (\widehat{Q}, \widehat{I}) \longrightarrow (Q, I)$ is then called a **covering** of bound quivers. In this situation, there is a normal subgroup H of $\pi_1(Q, I)$ such that $\pi_1(\widehat{Q}, \widehat{I}) \simeq H$, and $\pi_1(Q, I)/H \simeq G$. In particular, if $\pi_1(\widehat{Q}, \widehat{I}) =$

1, then $\pi_1(Q,I) \simeq G$ (see [7], for instance). Again, in this situation, the analogy with coverings of topological spaces is clear. It is shown in [6] that $\widehat{\mathcal{B}} = \mathcal{B}(\widehat{Q},\widehat{I})$ is a regular covering space of $\mathcal{B} = \mathcal{B}(Q,I)$ with $\mathrm{Cov}(\widehat{\mathcal{B}}/\mathcal{B}) \simeq G$.

Example. For $n \geq 2$, let $\widehat{Q}^n$ be the quiver whose vertices are $x_{i,j}$, with $0 \leq i < n$, $1 \leq j \leq n$ The arrows of $\widehat{Q}^n$ are $\alpha_{i,j} : x_{i,j} \longrightarrow x_{i,j-1}$ and $\beta_{i,j} : x_{i,j} \longrightarrow x_{i+1,j-1}$ for $0 \leq i \leq n$, $1 \leq j \leq n$, where indices are to be read modulo n. Moreover, let $\widehat{I} =< w - w' | s(w) = s(w'),\ t(w) = t(w') >$. Consider the automorphism $g : (\widehat{Q}^n, \widehat{I}) \longrightarrow (\widehat{Q}^n, \widehat{I})$ defined on the vertices of Q^n by $g(x_{i,j}) = x_{i+1,j}$, and on the arrows by $g(\alpha_{i,j}) = \alpha_{i+1,j}$, and $g(\beta_{i,j}) = \beta_{i+1,j}$. Then g has order n. We can form the quotient $(Q^n, I) = (\widehat{Q}^n, \widehat{I})/ < g >$. The vertex set of the quiver Q^n is given by: $Q^n_0 = \{x_0, x_1, \ldots, x_n\}$, and the arrows are $\alpha_j,\ \beta_j : x_j \longrightarrow x_{j-1}$, for $1 \leq j \leq n$. Moreover, $I \ =< \alpha_n \cdots \alpha_1 - \beta_n \cdots \beta_1,\ \alpha_i\beta_{i-1} - \beta_i\alpha_{i-1} |\ 1 < i \leq n >$. An immediate computation shows that $\pi_1(\widehat{Q}^n, \widehat{I}) = 1$ (see also [10, 5]), so that $\pi_1(Q^n, I) \simeq$ $< g > \simeq \mathbb{Z}_n$.

The quivers $\widehat{Q}^2$, and Q^2 look as follows:

$$\begin{tikzcd}
x_{0,2} \arrow[d, "\alpha_{0,2}"'] \arrow[dr, "\beta_{0,2}"] & x_{1,2} \arrow[d, "\alpha_{1,2}"] \arrow[dl, "\beta_{1,2}"'] \\
x_{0,1} \arrow[d, "\alpha_{0,1}"'] \arrow[dr, "\beta_{0,1}"] & x_{1,1} \arrow[d, "\alpha_{1,1}"] \arrow[dl, "\beta_{1,1}"'] \\
x_{0,0} & x_{1,0}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x_2 \arrow[d, bend right, "\alpha_2"'] \arrow[d, bend left, "\beta_2"] \\
x_1 \arrow[d, bend right, "\alpha_1"'] \arrow[d, bend left, "\beta_1"] \\
x_1
\end{tikzcd}$$

2. Coproducts and Products

Given two groups, say G_1 and G_2, one can consider at least two new groups, namely the free product $G_1 * G_2 = G_1 \amalg G_2$, and the direct product $G_1 \times G_2$. In this section, we show how to carry this constructions to fundamental groups of bound quivers. Again, the main ideas come from algebraic topology. On one hand Van Kampen's theorem tells that some fundamental groups of topological spaces are push-outs of groups, thus free products are involved. On the other hand, given two pointed topological spaces (X, x_0) and (Y, y_0) one can form the product $(X \times Y, (x_0, y_0))$, and then $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

2.1. Co-products. Given two pointed bound quivers $Q' = (Q', I', x')$ and $Q'' = (Q'', I'', x'')$, assume, without loss of generality, that $Q'_0 \cap Q''_0 = Q'_1 \cap Q''_1 = \emptyset$. We define the quiver $Q = Q' \amalg Q''$ in the following way: Q_0 is $Q'_0 \cup Q''_0$ in which we identify x' and x'' to a single new vertex x, and $Q_1 = Q'_1 \cup Q''_1$. Then, Q' and Q'' are identified to two full convex sub-quivers of Q, so walks on Q' or Q'' can be considered as walks on Q. Thus, I' and I'' generate two-sided ideals of kQ which we denote again by I' and I''. We define I to be the ideal $I' + I''$ of kQ. It follows from this definition that the minimal relations of I' together with the minimal relations of I'' give the minimal relations needed to determine the homotopy relation in (Q, I). In addition, we can consider an element $\widetilde{w} \in \pi_1(Q', I', x')$ as an element $\widetilde{w} \in \pi_1(Q, I, x)$. Conversely, any (reduced) walk $w \in W(Q, x)$ has a decomposition $w = w'_1 w''_1 w'_2 w''_2 \cdots w'_n w''_n$ where $w'_i \in W(Q', x')$, and $w''_i \in W(Q'', x'')$, for $i \in \{1, \ldots, n\}$, which is unique up to reduced walk. In addition, this decomposition is

compatible with the homotopy relations involved. This leads us to the following proposition.

Proposition. With the notations above we have:

i) (Q, I, x) is the coproduct, in the category of pointed bound quivers, of (Q', I', x) and (Q'', I'', x)

ii) $\pi_1(Q, I, x) \simeq \pi_1(Q', I', x') \amalg \pi_1(Q'', I'', x'')$.

Proof : The first statement follows from a direct computation, while the second follows immediately from the above discussion. □

Remark. It is worth to note that the canonical morphisms of pointed bound quivers $j' : (Q', I', x') \longrightarrow (Q, I, x)$ and $j'' : (Q'', I'', x'') \longrightarrow (Q, I, x)$ do not induce morphisms of $k-$algebras. If one wants to consider morphisms of $k-$algebras, the arrows must be *reversed*. One then has canonical projections of $k-$algebras $p' : kQ'/I' \longrightarrow kx$ and $p'' : kQ''/I'' \longrightarrow kx$. A natural question then is if the diagram

$$\begin{tikzcd} kQ/I \arrow[r] \arrow[d] & kQ'/I' \arrow[d] \\ kQ''/I'' \arrow[r] & kx \end{tikzcd}$$

is a pull-back of $k-$algebras. The answer is no. In [13] it was shown that the pull-back of $kQ'/I \longrightarrow kx \longleftarrow kQ''/I''$ is kQ/J where $J = I + < Q'_1 Q''_1 > + < Q''_1 Q'_1 >$.

We now turn our interest into direct products.

2.2. Products. As before, consider two pointed bound quivers (Q', I', x'), and (Q'', I'', x'') whose source maps and targets maps are s', s'', t', and t''. Following [12], we define the **product quiver** $Q = Q' \otimes Q''$ as follows. The vertex set Q_0 is simply $Q'_0 \times Q''_0$, whereas the arrow set is $Q_1 = (Q'_1 \times Q''_0) \cup (Q'_0 \times Q''_1)$. Given an arrow $(\alpha, y) \in Q'_1 \times Q''_0 \subseteq Q_1$, define $s(\alpha, y) = (s'(\alpha), y)$, and $t(\alpha, y) = (t'(\alpha), y)$. In an analogous way, we define $s(x, \beta)$ and $t(x, \beta)$ for an arrow $(x, \beta) \in Q'_0 \times Q''_1$. Now let $x = (x', x'')$ be the distinguished vertex in Q.

Given a vertex $y \in Q''_0$, a path $w = \alpha_1 \cdots \alpha_r$ in Q' induces a path $(\alpha_1, y) \cdots (\alpha_r, y)$ in Q, which we will denote by (w, y). Similarly, for every vertex y of Q', any path w in Q'' induces a path (y, w) in Q.

We define I to be the ideal whose generators are the following relations in kQ:

(a) (ρ, y), for every generator ρ of I', and every vertex $y \in Q''$,

(b) (x, ρ), for every generator ρ of I'', and every vertex $x \in Q'$,

(c) $(x_1, \beta)(\alpha, y_2) - (\alpha, y_1)(x_2, \beta)$ for every arrow $\alpha : x_1 \longrightarrow x_2$ in Q' and every arrow $\beta : y_1 \longrightarrow y_2$ in Q''.

$$\begin{tikzcd} (x_1, y_1) \arrow[r, "{(x_1,\beta)}"] \arrow[d, "{(\alpha,y_1)}"'] & (x_1, y_2) \arrow[d, "{(\alpha,y_2)}"] \\ (x_2, y_1) \arrow[r, "{(x_2,\beta)}"'] & (x_2, y_2) \end{tikzcd}$$

With these notations we have an isomorphism of $k-$algebras $kQ'/I'\otimes_k kQ''/I'' \simeq kQ/I$ (see [12]). However, note that since the natural projections from (Q, I, x) to (Q', I', x') and (Q'', I'', x'') *are not* morphisms of pointed bound quivers, the product quiver is not the product of (Q', I', x') and (Q'', I'', x'') in the category of pointed bound quivers.

Nevertheless, given an arrow Θ in Q, we can define the path $f'(\Theta)$ in Q' in the following way

$$f'(\Theta) = \begin{cases} \theta & \text{if } \Theta = (\theta, y) \in Q'_1 \times Q''_0, \\ e_x & \text{if } \Theta = (x, \theta) \in Q'_0 \times Q''_1. \end{cases}$$

Extending this map in the obvious way to walks in Q, we obtain a map $f' : W(Q, x) \longrightarrow W(Q', x')$ which is, in fact, a group homomorphism. In the same way, we obtain $f'' : W(Q, x) \longrightarrow W(Q'', x'')$. This leads us to the following lemma.

Lemma. *The maps f' and f'' defined above induce groups homomorphisms $\phi' : \pi_1(Q, I, x) \longrightarrow \pi_1(Q', I', x')$ and $\phi'' : \pi_1(Q, I, x) \longrightarrow \pi_1(Q'', I'', x'')$ given by the rules $\phi'(\widetilde{w}) = \widetilde{f'(w)}$ and $\phi''(\widetilde{w}) = \widetilde{f''(w)}$.*

Proof : The only thing one has to show is that ϕ' and ϕ'' are well defined maps. We do so only for ϕ'. For this sake, define the relation $\approx$ in $W(Q, x)$ by $w_1 \approx w_2$ if and only if $\widetilde{f'(w_1)} = \widetilde{f'(w_2)}$. Keeping in mind the generators of I, it is easily seen that this is an equivalence relation on $W(Q, x)$ which verifies the conditions (1), (2), and (3) of the definition of the homotopy relation on this set. Since the latter is the smallest such relation, we have that $w_1 \sim w_2$ implies $w_1 \approx w_2$, that is $w_1 \sim w_2$ implies $\phi'(\widetilde{w_1}) = \phi'(\widetilde{w_2})$. This shows that ϕ' is well defined. □

This leads us to the following proposition.

Proposition. *With the notations above, we have an isomorphism of groups*

$$\pi_1(Q, I, x) \simeq \pi_1(Q', I', x') \times \pi_1(Q'', I'', x'').$$

Proof : In light of the preceding lemma, we already have a morphism of groups $\Phi = (\phi', \phi'') : \pi_1(Q, I, x) \longrightarrow \pi_1(Q', I', x') \times \pi_1(Q'', I'', x'')$. In order to show that this is an isomorphism, we exhibit its inverse.

As noted before, given a walk w in Q', we can consider the walk (w, x'') in Q. This yields a map $\psi' : \pi_1(Q', I', x') \longrightarrow \pi_1(Q, I, x)$ defined by $\psi'(\widetilde{w}) = \widetilde{(w, x'')}$, which is, in fact a group homomorphism. In the same way we obtain a group homomorphism $\psi'' : \pi_1(Q'', I'', x'') \longrightarrow \pi_1(Q, I, x)$. This allows to define a group homomorphism $\psi = \psi' \amalg \psi'' : \pi_1(Q', I', x') \amalg \pi_1(Q'', I'', x'') \longrightarrow \pi_1(Q, I, x)$. Using relations of type c) in the definition of the generators of I, one can easily see that given $\widetilde{w_1} \in \pi_1(Q', I', x')$, and $\widetilde{w_2} \in \pi_1(Q'', I'', x'')$ one has $\psi(\widetilde{w_1}\widetilde{w_2}) = \psi(\widetilde{w_2}\widetilde{w_1})$. Thus, $\widetilde{w_1}\widetilde{w_2}\widetilde{w_1}^{-1}\widetilde{w_2}^{-1}$ belongs to Ker ψ, and, passing to the factor group, we can define a map $\Psi : \pi_1(Q', I', x') \times \pi_1(Q'', I'', x'') \longrightarrow \pi_1(Q, I)$ given by $\Psi(\widetilde{w_1}, \widetilde{w_2}) = \psi(\widetilde{w_1}\widetilde{w_2})$. Finally, it is a straightforward verification that Φ, and Ψ are mutually inverses. □

3. Changes of presentations

We have already encountered an example of an algebra A having two presentations (Q, I_1) and (Q, I_2) such that $\pi_1(Q, I_1) \simeq \mathbb{Z}$, and $\pi_1(Q, I_2)$ is trivial. Thus, we know how to pass from an infinite cyclic group to a trivial group. We begin this section by tackling the analogous question for a finite (non-trivial) cyclic group $\mathbb{Z}_n$.

Example 1. In section 1.3 we considered the quiver Q^n:

$$\begin{tikzcd} x_n \arrow[r, bend left, "\alpha_n"] \arrow[r, bend right, "\beta_n"'] & x_{n-1} \arrow[r, bend left, "\alpha_{n-1}"] \arrow[r, bend right, "\beta_{n-1}"'] & \cdots & \cdots \arrow[r, bend left, "\alpha_2"] \arrow[r, bend right, "\beta_2"'] & x_1 \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\beta_1"'] & x_0 \end{tikzcd}$$

bound by $I =< \alpha_n \cdots \alpha_1 - \beta_n \cdots \beta_1,\ \alpha_i\beta_{i-1} - \beta_i\alpha_{i-1} |\ 1 < i \leq n\ >$. Recall that $\pi_1(Q^n, I) \simeq \mathbb{Z}_n$.

Consider the presentation $\nu : kQ^n \longrightarrow kQ^n/I$ defined by $\nu(\alpha_n) = (\alpha_n - \beta_n) + I$, and $\nu(\gamma) = \gamma + I$ for every arrow $\gamma \in Q_1^n,\ \gamma \neq \alpha_n$. Let $J = \mathrm{Ker}\ \nu$. We pretend that $\pi_1(Q^n, J) \simeq 1$. In order to show this, let us begin by computing the minimal relations of J.

(1) Let $i \in \{2, \dots, n\}$. Clearly, $\nu(\alpha_i\beta_{i-1} - \beta_i\alpha_{i-1}) \in I$. Thus, $\alpha_i\beta_{i-1} - \beta_i\alpha_{i-1} \in J$, and this is a minimal relation.

(2) We pretend that $\alpha_n\beta_{n-1} + \beta_n\beta_{n-1} - \beta_n\alpha_{n-1}$ is a minimal relation. To show this, it suffices to show that its image by ν lies in I, since the minimality is clear. We have:

$$\begin{aligned} &\nu(\alpha_n\beta_{n-1} + \beta_n\beta_{n-1} - \beta_n\alpha_{n-1}) \\ &= \alpha_n\beta_{n-1} - \beta_n\beta_{n-1} + \beta_n\beta_{n-1} - \beta_n\alpha_{n-1} + I \\ &= I \end{aligned}$$

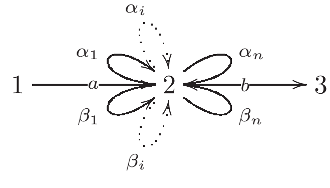
(3) We pretend that $\alpha_n \cdots \alpha_1 + \beta_n\alpha_{n-1} \cdots \alpha_1 - \beta_n \cdots \beta_1$ is a minimal relation. Indeed,

$$\begin{aligned} &\nu(\alpha_n \cdots \alpha_1 + \beta_n\alpha_{n-1} \cdots \alpha_1 - \beta_n \cdots \beta_1) \\ &= \alpha_n\alpha_{n-1} \cdots \alpha_1 - \beta_n\alpha_{n-1} \dots \alpha_1 + \beta_n\alpha_{n-1} \cdots \alpha_1 - \beta_n \cdots \beta_1 + I \\ &= I. \end{aligned}$$

The relation 2, above, gives us $\alpha_n \sim \beta_n$, and $\alpha_{n-1} \sim \beta_{n-1}$. Moreover, letting $i = n-1$ in relations of type 2, we get $\alpha_{n-2} \sim \beta_{n-2}$. The same argument, decreasing the value of i gives $\alpha_j \sim \beta_j$ for $j \neq 1$. Finally, relation 3 gives $\alpha_1 \sim \beta_1$. This shows that $\pi_1(Q^n, J) = 1$.

The following example illustrates how changes of presentations can be done with quivers having loops, More precisely, it shows how one can pass from any finitely presented group G to the trivial group.

Example 2. Let G be a finitely presented group, (Q_G, I_G) the bound quiver of example 2, in section 1.2. Moreover, set $A = kQ_G/I_G$.



Recall that $I_G =< a\alpha_i\beta_i b - ab,\ aw_j b - ab, F^N | 1 \leq i \leq n, 1 \leq j \leq m\}$ where $N = \max\{l(w_j) + 3, 6 | 1 \leq j \leq m\}$, and this leads to $\pi_1(Q_G, I_G) \simeq G$.

Consider now the morphism $\nu : kQ_G \longrightarrow kQ_G/I_G$ defined on the arrows of Q_G by

$$\nu(\gamma) = \begin{cases} \alpha_i + \beta_i + \beta_i^2 + I_G & \text{if } \gamma = \alpha_i, \\ \gamma + I_G & \text{otherwise.} \end{cases}$$

Since $\{\nu(\alpha_i) + \mathrm{rad}^2 A, \nu(\beta_i) + \mathrm{rad}^2 A | 1 \leq i \leq n\} = \{\alpha_i + \beta_i + \mathrm{rad}^2 A, \beta_i + \mathrm{rad}^2 A\}$, this set is a basis of $e_2(\mathrm{rad}A/\mathrm{rad}^2 A)e_2$, so that ν is a presentation of A. In particular Ker ν is an admissible ideal. We claim that $\pi_1(Q_G, \mathrm{Ker}\ \nu) \simeq 1$.

We show that $\rho = a\alpha_i\beta_i b - a\beta_i^2 b - a\beta_i^3 b - ab$ is a minimal relation. Indeed

$$\begin{aligned} \nu(\rho) &= \nu(a\alpha_i\beta_i b - a\beta_i^2 b - a\beta_i^3) \\ &= a(\alpha_i + \beta_i + \beta_i^2)\beta_i b - a\beta_i^2 b - a\beta_i^3 b - ab + I_G \\ &= a\alpha_i\beta_i b - ab + I_G \\ &= I_G \end{aligned}$$

Thus, it only remains to prove the minimality of ρ, but this follows from the following facts :

(1) $\nu(ab) = ab + I_G \neq I_G$,
(2) A linear combination of $a\beta_i^2 b$ and $a\beta_i^3 b$ cannot belong to I_G. Indeed, it follows from the hypothesis made on the words w_i that each one of them contains at least one α_i, thus is different from β_i^2 and β_i^3. In addition we have

$$l(a\beta_i^2 b) < l(a\beta_i^3 b) = 5 < N.$$

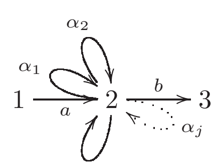
So we are done, and ρ is minimal. This yields

$$a\alpha_i\beta_i b \sim a\beta_i^2 b \sim a\beta_i^3 \sim ab$$

so that $\alpha_i\beta_i \sim e_2$, and $\beta_i^2 \sim \beta_i^3$, and this shows our claim.

The encountered examples all show that there are algebras having different presentations which have an arbitrary group, as well as the trivial group as fundamental groups. The following example shows how one can pass directly from any finitely generated abelian group to a free abelian group.

Example 3. Fix a positive integer t, and, for each $i \in \{1, \ldots, t\}$, let $n_i \in \mathbb{N}_*$, and let $m > \max_i\{n_i\} + 1$. Consider the quiver



bound by $I =< a\alpha_i b + a\alpha_i^{n_i+1} b,\ \alpha_i\alpha_j - \alpha_j\alpha_i, F^m |\ 1 \leq i \leq t >$. A direct computation shows that $\pi_1(Q, I) \simeq \bigoplus_{i=1}^t \mathbb{Z}_{n_i}$. On the other hand, consider the presentation $\nu : kQ \longrightarrow kQ/I$ given by $\nu(\alpha_i) = (\alpha_i - \alpha_i^{n_i+1}) + I$. We leave the reader verify that $I' = \mathrm{Ker}\ \nu =< a\alpha_i b, \alpha_i\alpha_j - \alpha_j\alpha_i, F^m |\ 1 \leq i, j \leq t >$, and this leads to $\pi_1(Q, I') = \bigoplus_{i=1}^t \mathbb{Z}$.

Note that choosing $n_i = 1$ for all $i \in \{1, \dots, t\}$ we obtain that $\pi_1(Q, I)$ is the trivial group.

In section 2, we saw that fundamental groups of bound quivers behave well under products and coproducts. The following lemma shows that the same is true under changes of presentations.

Lemma. *For $i \in \{1, 2\}$, let $A_i \simeq kQ_{A_i}/I_i \simeq kQ_{A_i}/I'_i$ be algebras with two different presentations. Denote by $\pi_1(Q_{A_i}, I_i) \simeq G_i$, and $\pi_1(Q_{A_i}, I'_i) \simeq G'_i$. Then we have the following:*

i) There exists an algebra A having two presentations $A \simeq kQ/I \simeq kQ/I'$ such that $\pi_1(Q, I) \simeq G_1 \amalg G_2$ and $\pi_1(Q, I') \simeq G'_1 \amalg G'_2$.

ii) There exists an algebra A having two presentations $A \simeq kQ/I \simeq kQ/I'$ such that $\pi_1(Q, I) \simeq G_1 \times G_2$ and $\pi_1(Q, I') \simeq G'_1 \times G'_2$.

Proof : For $i \in \{1, 2\}$, let $\nu_i : kQ_{A_i} \longrightarrow A_i$, and $\nu'_i : kQ_{A_i} \longrightarrow A_i$ be the presentations of the algebra A_i such that $I_i = \text{Ker } \nu_i$, and $I'_i = \text{Ker } \nu'_i$. In order to prove the first statement, consider the pointed bound quiver $(Q, I, x) = (Q_{A_1}, I_1, x_1) \amalg (Q_{A_2}, I_2, x_2)$ and let $A = kQ/I$. It follows from theorem 2.1 that $\pi_1(Q, I, x) \simeq G_1 \amalg G_2$. On the other hand, consider the presentation $\nu' : kQ \longrightarrow kQ/I \simeq A$ given by $\nu'(\alpha) = \nu'_i(\alpha)$, where α is an arrow of Q_{A_i}. Then, we have $I' = \text{Ker } \nu' = I'_1 + I'_2$, and, again, from theorem 2.1, we obtain $\pi_1(Q, I', x) \simeq G'_1 \amalg G'_2$.

In order to prove the second statement, consider the quiver $Q = Q_{A_1} \otimes Q_{A_2}$ bound by the ideal I as described in section 2.2. We then have, from theorem 2.2, that $\pi_1(Q, I, x) \simeq G_1 \times G_2$. Let $A = kQ/I$, and consider the following presentation $\nu' : kQ \longrightarrow kQ/I \simeq A$ given by

$$\nu'(\Theta) = \begin{cases} \nu'_1(\theta) & \text{if } \Theta = (\theta, y) \in (Q_{A_1})_1 \times (Q_{A_2})_0, \\ \nu'_2(\theta) & \text{if } \Theta = (x, \theta) \in (Q_{A_1})_0 \times (Q_{A_2})_1. \end{cases}$$

Again, it follows from the definition of the ideal I, and using the fact that ν'_1 and ν'_2 are changes of presentations, that $I = \text{Ker } \nu'$. Thus, theorem 2.2 gives $\pi_1(Q, I', x) \simeq G'_1 \times G'_2$. □

We are now able to prove our main results.

4. Main Results

Theorem A. *Let $G_1, \dots, G_n$ be finitely presented groups. Then, there exists a finite dimensional algebra A having presentations $A \simeq kQ_A/I_i$, for $i \in \{1, \dots, n\}$, such that $\pi_1(Q_A, I_i) \simeq G_i$.*

Proof : Using example 3.2, we can build algebras A_i having presentations $A_i \simeq kQ_i/J_i \simeq kQ_i/J'_i$ with $\pi_1(Q_i, J_i) \simeq G_i$, and $\pi_1(Q_i, J'_i) \simeq 1$. For $i \in \{1, \dots, n\}$ consider the bound quiver

$$(Q_A, I_i) = \left(\coprod_{l=1}^{i-1} (Q_l, J'_l) \right) \amalg (Q_i, J_i) \amalg \left(\coprod_{l=i+1}^{n} (Q_l, J'_l) \right).$$

It follows from theorem 2.1, statement *ii)* that $\pi_1(Q_A, I_i) \simeq G_i$. Moreover, using the argument of the proof of the above lemma, one gets $kQ_A/I_i \simeq kQ_A/I_j$, for all $i, j \in \{1, \dots, n\}$. □

The examples of quivers without oriented cycles that we have encountered led us to consider cyclic groups. Moreover, from results in section 2 we know how to deal with products and coproducts of fundamental groups. Let us denote by $\mathbb{G}$ the smaller family of groups satisfying the following conditions :

(1) If G is a cyclic group then $G \in \mathbb{G}$,
(2) If G_1, G_2 belong to $\mathbb{G}$, then the same holds for $G_1 \coprod G_2$ and $G_1 \times G_2$.

This leads us to the following theorem.

Theorem B. *Let $G_1, \ldots, G_n \in \mathbb{G}$. Then, there exists a triangular algebra A having presentations $A \simeq kQ_A/I_i$, for $i \in \{1, \ldots, n\}$, such that $\pi_1(Q_A, I_i) \simeq G_i$.*

Proof : Using propositions 2.1, 2.2, example 1 in section 3, example 1 in 1.2, and the above lemma, we can build triangular algebras A_i having presentations $A_i \simeq kQ_i/J_i \simeq kQ_i/J'_i$ with $\pi_1(Q_i, J_i) \simeq G_i$, and $\pi_1(Q_i, J'_i) \simeq 1$. The remaining part of the proof is just as in the preceding result. □

Corollary. Let $M_1, \ldots, M_n$ be finitely generated abelian groups. Then there exists a triangular algebra A having presentations $A \simeq kQ_A/I_i$, for $i \in \{1, \ldots, n\}$, such that $\pi_1(Q_A, I_i) \simeq M_i$. □

As a final remark, let us note that the homotopy relation in a bound quiver does make sense even if we do not ask the ideal I to be admissible, nor the quiver Q to be finite. These requirements lead to finite dimensional algebras (with 1).

If one wishes to consider the family of all finitely generated groups, similar constructions of what have been made in this work can be done. Indeed given any finitely generated group G, as in Example 2 of 1.2 one can construct a quiver Q_G and a two sided ideal $I_G \trianglelefteq kQ_G$ such that $\pi_1(Q_G, I_G) \simeq G$. In this case, the quiver would still be finite, but the ideal I will not be admissible, so this would lead to infinite dimensional algebras which can be seen as locally bounded $k-$categories.

As further generalisation, one may wish to consider arbitrary groups. Again, this can be performed, as in Example 2 of 1.2. This time the quiver will still have three vertices, but will have an infinite number of arrows, and, again, the ideal would not be admissible.

Also, in light of Bardzell-Marcos theorem, one may ask how big is the family of groups that rise as fundamental groups of presentation of a fixed algebra A.

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